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FINITE GROUPS WITH MANY PRODUCT CONJUGACY CLASSES

BY

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ABSTRACT

We classify all finite groups G such that the product of any two non-inverse conjugacy classes of G is always a conjugacy class of G. We also classify all finite groups G for which the product of any two G-conjugacy classes which are not inverse modulo the center of G is again a conjugacy class of G.

1. Introduction

We're going to consider finite groups G in which the product of two conjugacy classes is itself a conjugacy class, except in a few cases where that product obviously can't be a single conjugacy class. If we denote by x^G the G-conjugacy class of an element $x \in G$, then the product $x^G y^G$ of two conjugacy classes x^G and y^G in G is itself a conjugacy class if and only if it satisfies

$$x^G y^G = (xy)^G. \tag{1.1}$$

This equation holds trivially if either x or y belongs to the center $\mathrm{Z}(G)$ of G. So we may assume that both x and y lie in the complementary subset $G - \mathrm{Z}(G)$

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to $\mathrm{Z}(G)$ in G. Then (1.1) certainly fails when x^G is the inverse class $(y^{-1})^G$ to y^G, since $x^G y^G$ then contains the trivial conjugacy class $1^G = \{1\}$, but has size $|x^G y^G| \geq |x^G| > 1$, and thus must contain at least one other conjugacy class. Our first assumption is that this is the only situation in which (1.1) fails. So we are going to consider finite groups G satisfying

HYPOTHESIS A: *Equation (1.1) holds for all $x, y \in G$ such that $x^G \neq (y^{-1})^G$.*

Of course, this hypothesis holds for every abelian finite group G. Another large class of groups satisfying it consists of the Camina p-groups. These are the finite p-groups G such that each non-trivial coset $x[G,G]$ of the derived group $[G,G]$ is a single conjugacy class x^G in G. Of the many papers about such groups we only mention [2], [5] and [6]. Using some of the deepest results in those papers, we can show that all Camina p-groups satisfy Hypothesis A.

Hypothesis A also holds for a few other groups G. One such group is the semi-direct product $F^+ \rtimes F^\times$ of the additive group F^+ of any finite field F with the multiplicative group $F^\times$ of F. This semi-direct product is just cyclic of order 2 when $|F| = 2$. But it is a Frobenius group for all other $|F|$. Another Frobenius group G satisfying Hypothesis A is the semi-direct product $E_9 \rtimes Q_8$ of an elementary abelian group E_9 of order 9 with the unique quaternion subgroup Q_8 of order 8 in the automorphism group of E_9. The interest of $E_9 \rtimes Q_8$ as an example was pointed out by Camina in his original paper [1].

Our first main result is that the above groups are the only ones.

THEOREM A: *A finite group G satisfies Hypothesis A if and only if it is isomorphic to exactly one of the groups in the following list:*

(1.2a) *Any finite abelian group.*

(1.2b) *A non-abelian Camina p-group, for some prime p.*

(1.2c) *The group $F^+ \rtimes F^\times$, for some finite field F with $|F| > 2$.*

(1.2d) *The group $E_9 \rtimes Q_8$.*

Another situation where (1.1) clearly fails is when the classes x^G and $(y^{-1})^G$ have the same non-trivial image $x^G\mathrm{Z}(G)/\mathrm{Z}(G) = (y^{-1})^G\mathrm{Z}(G)/\mathrm{Z}(G) \neq 1_{G/\mathrm{Z}(G)}$ in the factor group $G/\mathrm{Z}(G)$. In this case $x^G y^G$ contains some element $z \in \mathrm{Z}(G)$, yet has size $|x^G y^G| \geq |x^G| > 1$. So it contains both $z^G = \{z\}$ and at least one other conjugacy class of G. Weakening Hypothesis A to avoid this situation, we obtain

HYPOTHESIS B: *If $x, y \in G$ and $x^G\mathrm{Z}(G) \neq (y^{-1})^G\mathrm{Z}(G)$, then (1.1) holds.*

When a finite group G satisfies this weaker hypothesis, so does any finite group H isoclinic to G, in the sense of Philip Hall [3] (see Proposition 4.12 below). Our other main result is that this is the only freedom we have.

THEOREM B: *A finite group G satisfies Hypothesis B if and only if it is isoclinic to a group satisfying Hypothesis A, and thus to one of the groups on the list (1.2).*

Since all abelian groups satisfy both our hypotheses, Theorem A follows from Theorems 4.3 and 5.8 below, while Theorem B follows from Theorems 4.13 and 5.11. Another consequence of Theorems 4.3 and 5.8 is

THEOREM C: *A non-abelian finite group G satisfying Hypothesis B satisfies the stronger Hypothesis A if and only if $\mathrm{Z}(G) \leq [G,G]$.*

2. Notation

Our notation for objects associated with a finite multiplicative group G is mostly standard. We use 1 or 1_G to denote both the identity element of G and the trivial subgroup $\{1\}$ of G. We write $\langle x\rangle$ for the cyclic subgroup of G generated by a given element $x \in G$. We denote by $G^{\#}$ the set $G-1 = G-\{1\}$ of all non-identity elements of G. To say that some H is a subset, a subgroup or a normal subgroup of G we write $H \subseteq G$, $H \leq G$ or $H \trianglelefteq G$, respectively. To indicate, in addition, that H is properly contained in G, we write $H \subsetneq G$, $H < G$ or $H \triangleleft G$, respectively. If $x, y \in G$, then x^y denotes the conjugate element $y^{-1}xy \in G$ and $[x,y] = [x,y]_G$ denotes the commutator $x^{-1}y^{-1}xy = x^{-1}x^y \in G$. If $x \in G$ and $H \leq G$, then x^H denotes the H-conjugacy class of all x^w, for $w \in H$, and $[x,H]$ denotes the set of all $[x,w]$, for $w \in H$. Since $x^w = x[x,w]$ for all $w \in H$, we have $x^H = x[x,H]$. It follows that $x^Hy^H = x[x,H]y[y,H] = xy[x,H]^y[y,H]$, and $(xy)^H = xy[xy,H]$, for any $x, y \in G$. Hence

$$x^Hy^H = (xy)^H \quad \text{if and only if } [x,H]^y[y,H] = [xy,H] \tag{2.1}$$

for any $x, y \in G$ and $H \leq G$.

By long-standing convention the expression $[K,H]$, for subgroups $K, H \leq G$, denotes, not the set of all commutators $[x,y]$, for $x \in K$ and $y \in H$, but rather the subgroup $\langle [x,y] \mid x \in K, y \in H\rangle$ of G generated by those commutators. This does not conflict with the previous notation when K is the element or subgroup 1 of G, since both the subset $[1,H]$ and the subgroup $[1,H]$ are equal to $\{1\}$. However, there could be a conflict when $K = H = N$, for some $N \trianglelefteq G$. In this case N is both a subgroup of G and the identity element $1_{G/N}$ of the factor group

G/N. So the expression $[N, N]$, when defined using commutation in G, denotes the commutator subgroup of N, while that same expression, when defined using commutation in G/N, denotes the identity element $N = 1_{G/N} = [1_{G/N}, 1_{G/N}]$ of G/N. To avoid this ambiguity, we always write any commutator expression in the form $[X, Y]_{G/N}$ when it is to be computed using commutators in G/N, reserving the notation $[X, Y]$ for expressions to be computed in G.

We write the subgroups in the lower central series of G as $\gamma_n(G)$, where n runs over all strictly positive integers. They are defined inductively by

$$\gamma_1(G) = G \tag{2.2a}$$

and

$$\gamma_{n+1}(G) = [\gamma_n(G), G] \tag{2.2b}$$

for any integer $n \geq 1$. Note that $\gamma_2(G)$ is the derived group $[G, G]$ of G. Because G is finite, there is a unique integer $c \geq 0$ such that $\gamma_1(G) > \gamma_2(G) > \cdots > \gamma_{c+1}(G) = \gamma_{c+2}(G) = \cdots$. We define $\gamma_\infty(G)$ to be the "limit group" $\gamma_{c+1}(G)$. So the full lower central series for G is

$$G = \gamma_1(G) > \gamma_2(G) > \cdots > \gamma_{c+1}(G) = \gamma_{c+2}(G) = \cdots = \gamma_\infty(G). \tag{2.3}$$

It follows from this and (2.2b) that

$$[\gamma_\infty(G), G] = \gamma_\infty(G). \tag{2.4}$$

Furthermore, the factor group $G/\gamma_\infty(G)$ is nilpotent with class c. Indeed, $\gamma_\infty(G)$ is the smallest normal subgroup N of G such that the factor group G/N is nilpotent.

If $H \leq G$, and X is either a subset or an element of G, then $\mathrm{N}_H(X)$ and $\mathrm{C}_H(X)$ denote the normalizer and centralizer, respectively, of X in H. We write $\mathrm{Z}(G)$ for the center $\mathrm{C}_G(G)$ of G.

Our one non-standard notation concerns what we shall call the **right multiplier**

$$\mathrm{M}_H(S) = \{y \in H \mid Sy = S\} \tag{2.5}$$

of any subset $S \subseteq G$ in any subgroup $H \leq G$. Clearly $\mathrm{M}_H(S)$ is a subgroup of H, and S is a union of cosets $x\mathrm{M}_H(S)$ of this subgroup. Hence $|\mathrm{M}_H(S)|$ divides $|S|$. Because our group G is finite, so is its subset S. Since $|Sy| = |S|$, for any $y \in H$, we conclude that $Sy = S$ if and only if $Sy \subseteq S$. So we have the alternative definition

$$\mathrm{M}_H(S) = \{y \in H \mid Sy \subseteq S\} \tag{2.6}$$

for right multipliers in finite groups. Finally, we remark that the subgroup $\mathrm{M}_H(S)$ is normal in H whenever the subset S is H-invariant, in the usual sense that $S^x = S$ for all $x \in H$. In particular, $\mathrm{M}_G(x^G)$ is a normal subgroup of G, for any conjugacy class x^G in G.

3. Conjugacy classes

Our hypotheses pass immediately to factor groups.

PROPOSITION 3.1: *If a finite group G satisfies Hypothesis B, then so does the factor group G/N of G by any $N \trianglelefteq G$. If, in addition, N contains $\mathrm{Z}(G)$, then G/N satisfies the stronger Hypothesis A. On the other hand, G/N also satisfies Hypothesis A whenever G does.*

Proof: Let $\bar{G}$ be the factor group G/N. The natural epimorphism $e\colon x \mapsto xN$ of G onto $\bar{G}$ sends $\mathrm{Z}(G)$ into $\mathrm{Z}(\bar{G})$. It also sends the G-conjugacy class x^G of any $x \in G$ onto the $\bar{G}$-conjugacy class $e(x)^{\bar{G}}$ of $e(x)$. Hence it sends $x^G\mathrm{Z}(G)$ into $e(x)^{\bar{G}}\mathrm{Z}(\bar{G})$.

Suppose that $\bar{x}, \bar{y} \in \bar{G}$ satisfy $\bar{x}^{\bar{G}}\mathrm{Z}(\bar{G}) \neq (\bar{y}^{-1})^{\bar{G}}\mathrm{Z}(\bar{G})$. Then there are some elements $x, y \in G$ such that $\bar{x} = e(x)$ and $\bar{y} = e(y)$. If $x^G\mathrm{Z}(G) = (y^{-1})^G\mathrm{Z}(G)$, then

$$\bar{x}^{\bar{G}}\mathrm{Z}(\bar{G}) = e(x^G\mathrm{Z}(G))\mathrm{Z}(\bar{G}) = e((y^{-1})^G\mathrm{Z}(G))\mathrm{Z}(\bar{G}) = (\bar{y}^{-1})^{\bar{G}}\mathrm{Z}(\bar{G}),$$

which is false. So $x^G\mathrm{Z}(G) \neq (y^{-1})^G\mathrm{Z}(G)$. Hence $x^G y^G = (xy)^G$ by Hypothesis B for G. Applying the epimorphism e to this last equation, we obtain $\bar{x}^{\bar{G}}\bar{y}^{\bar{G}} = (\bar{x}\bar{y})^{\bar{G}}$. Therefore Hypothesis B holds for $\bar{G} = G/N$ whenever it holds for G.

Suppose that the above $\bar{x}, \bar{y} \in \bar{G}$ only satisfy $\bar{x}^{\bar{G}} \neq (\bar{y}^{-1})^{\bar{G}}$, but that $\mathrm{Z}(G) \leq N$. Then $e(\mathrm{Z}(G)) = 1_{\bar{G}}$, so that the equation $x^G\mathrm{Z}(G) = (y^{-1})^G\mathrm{Z}(G)$ still implies a contradiction $\bar{x}^{\bar{G}} = e(x^G\mathrm{Z}(G)) = e((y^{-1})^G\mathrm{Z}(G)) = (\bar{y}^{-1})^{\bar{G}}$. With this modification, the above argument shows that $\bar{G} = G/N$ satisfies Hypothesis A when $\mathrm{Z}(G) \leq N$ and G satisfies Hypothesis B.

Finally, if G satisfies Hypothesis A, then $\bar{x}^{\bar{G}} \neq (\bar{y}^{-1})^{\bar{G}}$ implies $x^G \neq (y^{-1})^G$, and thus implies both $x^G y^G = (xy)^G$ and $\bar{x}^{\bar{G}}\bar{y}^{\bar{G}} = (\bar{x}\bar{y})^{\bar{G}}$. So $\bar{G}$ satisfies Hypothesis A, and the proposition is proved. ∎

A simple, but useful, consequence of Hypothesis A is

PROPOSITION 3.2: *If a finite group G satisfies Hypothesis A, then $x^G\mathrm{Z}(G) = x^G$ for every element $x \in G - \mathrm{Z}(G)$. Hence $\mathrm{Z}(G) \leq [K, G]$ whenever K is a non-central subgroup of G.*

Proof: The product $x^G z$ is equal to the class $(xz)^G = ((x^{-1}z^{-1})^{-1})^G$, for any $x \in G - \mathrm{Z}(G)$ and any $z \in \mathrm{Z}(G)$. If $x^G \neq x^G z$, then Hypothesis A for $y = x^{-1}z^{-1}$ tells us that $x^G(x^{-1}z^{-1})^G = (x(x^{-1}z^{-1}))^G = (z^{-1})^G = \{z^{-1}\}$. This is impossible because $|x^G| > 1$. Therefore $x^G z = x^G$ for all $z \in \mathrm{Z}(G)$, i.e., $x^G\mathrm{Z}(G) = x^G$.

If K is a non-central subgroup of G, then there is some element $x \in K - \mathrm{Z}(G)$. By the above arguments we have $x^G\mathrm{Z}(G) = x^G$. Hence

$$\mathrm{Z}(G) = [x, 1]\mathrm{Z}(G) \subseteq [x, G]\mathrm{Z}(G) = x^{-1}x^G\mathrm{Z}(G) = x^{-1}x^G = [x, G] \subseteq [K, G].$$

So the proposition holds. ∎

For the rest of this section G will be an arbitrary finite group satisfying Hypothesis B. We want a clearer description of the possible conjugacy classes $x^G = x[x, G]$, for $x \in G$. Suppose that x lies in some normal subgroup N of G containing $\mathrm{Z}(G)$. Then $[N, G]$ is a normal subgroup of G contained in N. So is the product $[N, G]\mathrm{Z}(G)$. When x does not lie in $[N, G]\mathrm{Z}(G)$, the conjugacy class x^G has a very simple description.

PROPOSITION 3.3: *If $\mathrm{Z}(G) \leq N \trianglelefteq G$, for some finite group G satisfying Hypothesis B, then $[x, G] = [N, G]$ for all $x \in N - [N, G]\mathrm{Z}(G)$. Hence the conjugacy class $x^G = x[x, G]$ is the coset $x[N, G]$, for all such x.*

Proof: Let M be the normal subgroup $[N, G]\mathrm{Z}(G)$ of G contained in N. We prove the proposition in a series of steps.

STEP 1: *For any $x \in N - M$ the subset $[x, G]$ is a normal subgroup of G.*

Proof: We first show that $[x, G]y \subseteq [x, G]$, for any $y \in [x, G]$. Since x does not lie in the normal subgroup M of G, while y does, the conjugacy classes x^G and $(y^{-1})^G$ are subsets of the disjoint sets $N - M$ and M, respectively. Because M contains $\mathrm{Z}(G)$, it follows that $x^G\mathrm{Z}(G) \neq (y^{-1})^G\mathrm{Z}(G)$. Thus $x^G y^G = (xy)^G$ by Hypothesis B. But $y = [x, z] = x^{-1}x^z$, for some $z \in G$. Hence $xy = x^z$ is conjugate to x, and $(xy)^G = x^G$. Therefore

$$x[x, G]y = x^G y \subseteq x^G y^G = (xy)^G = x^G = x[x, G].$$

So $[x, G]y \subseteq [x, G]$, for all $y \in [x, G]$.

We have seen that the subset $[x, G]$ of the finite group G is closed under multiplication in G. Since $[x, G]$ contains $1 = [x, 1]$, it must be a subgroup of G. The commutator identity $[x, zw] = [x, w][x, z]^w$, for $w, z \in G$, implies that $[x, G]^w = [x, w]^{-1}[x, G] = [x, G]$, for any $w \in G$. So the subgroup $[x, G]$ is normal in G, and this step is proved. ∎

The relations between the various subgroups $[x, G]$ are based on

STEP 2: *If $x \in N - M$ and $y \in N$ satisfy $xM \neq y^{-1}M$, then $[x, G][y, G] = [xy, G]$.*

Proof: The conjugacy class x^G is $x[x, G] \subseteq x[N, G]$. Hence the product $x^G \mathrm{Z}(G)$ is contained in $x[N, G]\mathrm{Z}(G) = xM$. Similarly, $(y^{-1})^G$ is contained in $y^{-1}M$. Since the cosets xM and $y^{-1}M$ of M are different, they are disjoint. Therefore $x^G\mathrm{Z}(G) \neq (y^{-1})^G\mathrm{Z}(G)$. So $x^G y^G = (xy)^G$ by Hypothesis B. In view of (2.1) this last equation is equivalent to $[x, G]^y[y, G] = [xy, G]$. But $[x, G]^y = [x, G]$, since $[x, G] \trianglelefteq G$ by Step 1. Thus the present step holds. ∎

One consequence of the preceding step is

STEP 3: *For any $x \in N - M$, the normal subgroup $[x, G]$ contains $[M, G]$. So the subgroup $[x, G]$ depends only on the coset xM.*

Proof: If $y \in M$, then $xM \neq M = y^{-1}M$. Hence $[x, G][y, G] = [xy, G]$ by Step 2. It follows that $|[x, G]| \leq |[xy, G]|$. A similar argument, using $xy \in N - M$ and $y^{-1} \in M$ in place of x and y, respectively, shows that $|[xy, G]| \leq |[(xy)y^{-1}, G]| = |[x, G]|$. Therefore $|[xy, G]| = |[x, G]|$.

For any $z \in [y, G]$, the product $[x, G]z$ is a subset of size $|[x, G]|$ in the finite set $[x, G][y, G] = [xy, G]$ with the same size. So it must equal the latter set. Since $1 = [xy, 1]$ lies in $[xy, G]$, we conclude that z^{-1} belongs to the finite group $[x, G]$. It follows that $z \in [x, G]$ for all $z \in [y, G]$ and all $y \in M$. Thus $[M, G] \leq [x, G]$. This implies that $[xy, G] = [x, G][y, G] = [x, G]$, for all $y \in M$. Hence the subgroup $[x, G]$ depends only on the coset xM, and the step is proved. ∎

Now we can finish the proof of Proposition 3.3. Suppose that $x, y \in N - M$ satisfy $xM \neq yM$. Then $xy^{-1} \in N - M$, so that $[x, G]$, $[y, G]$ and $[xy^{-1}, G]$ are normal subgroups of G by Step 1. Furthermore, $xy^{-1}M = xMy^{-1}M$ is different from $y^{-1}M$, since $xM \neq M$. So Step 2, with xy^{-1} in place of x, tells us that $[xy^{-1}, G][y, G] = [(xy^{-1})y, G] = [x, G]$. Hence the subgroup $[y, G]$ is contained in $[x, G]$. By symmetry $[x, G]$ is contained in $[y, G]$. Thus $[x, G] = [y, G]$ whenever $x, y \in N - M$ lie in different cosets xM, yM of M. On the other hand, $[x, G] = [y, G]$ by Step 3 whenever $x, y \in N - M$ lie in the same coset $xM = yM$ of M. So the subgroup $[x, G]$ is independent of the choice of $x \in N - M$. Since this subgroup contains $[M, G]$ by Step 3, we conclude that it equals the subgroup $[N, G]$ generated by the subsets $[z, G]$ for $z \in N$. Thus Proposition 3.3 is proved. ∎

The other situation we must handle is that in which x belongs to some normal subgroup $N \trianglelefteq G$ such that $[N, G] = N$. For the moment we only treat the case where N is a minimal normal subgroup of G, i.e., is minimal among the non-trivial normal subgroups of G.

Proposition 3.4: *Let N be a minimal normal subgroup of some finite group G satisfying Hypothesis B. Suppose that $[N, G] = N$. Then $N^\# = N - 1$ is a single G-conjugacy class with size $|N^\#| > 1$. Hence N is an elementary abelian p-group, for some prime p.*

Proof: The intersection $N \cap \mathrm{Z}(G)$ is a normal subgroup of G contained in N. It cannot equal N, since $[N, G] = N > 1 = [N \cap \mathrm{Z}(G), G]$. So it must be 1 by the minimality of N. It follows that two elements $x, y \in N$ satisfy $x^G \mathrm{Z}(G) \neq (y^{-1})^G \mathrm{Z}(G)$ if and only if $x^G \neq (y^{-1})^G$. Thus Hypothesis B implies that

$$x^G y^G = (xy)^G \quad \text{whenever } x, y \in N \text{ and } x^G \neq (y^{-1})^G. \tag{3.5}$$

Our minimal normal subgroup N is non-trivial by definition. So there is some element $x \in N^\#$. Since $N \trianglelefteq G$, it follows that $x^G \subseteq N^\#$.

Suppose that $x^G \subsetneq N^\#$. Then some element $y \in N^\#$ satisfies $x^G \neq (y^{-1})^G$. By (3.5) this implies that $x^G y^G = (xy)^G$.

If $(xy)^G \neq y^G$, then we can apply (3.5) with xy and y^{-1} in place of x and y, respectively. It tells us that $(xy)^G (y^{-1})^G = x^G$. So $x^G y^G (y^{-1})^G = x^G$. Thus $y^G (y^{-1})^G$ is a subset of the group $\mathrm{M}_G(x^G)$ in (2.6). The multiplier $\mathrm{M}_G(x^G)$ is a normal subgroup of G. It is properly contained in N, since $x^G \subsetneq N$ is a non-empty union of cosets of $\mathrm{M}_G(x^G)$. So it must be 1 by the minimality of N. Thus $y^G (y^{-1})^G = 1$, which forces y^G to consist of a single element $y \in \mathrm{Z}(G)$. This is impossible, since $y \in N^\#$ and $N \cap \mathrm{Z}(G) = 1$. Therefore we must have $(xy)^G = y^G$.

Now $y^G x^G = x^G y^G = (xy)^G = y^G$. So x^G is a subset of $\mathrm{M}_G(y^G)$. But $\mathrm{M}_G(y^G)$, like $\mathrm{M}_G(x^G)$, must be 1. Therefore $x^G = 1$, which is impossible since $x \in N^\#$. Because the assumption $x^G \subsetneq N^\#$ always leads to a contradiction, we conclude that $N^{\#} = x^G$ is a single G-conjugacy class. Of course, $|N^{\#}| > 1$, since otherwise N would be central in G, contrary to our assumptions.

If the minimal normal subgroup N is not abelian, then it is a direct product of non-abelian simple groups. So its order must be divisible by at least two distinct primes p and q. Then $N^{\#}$ contains at least two distinct G-conjugacy classes — one consisting of elements with order p, and one consisting of elements with order q. This is false because $N^{\#}$ is a single conjugacy class. Thus N is abelian. The minimality of N now forces it to be an elementary abelian p-group, for some prime p. So the proposition holds. ∎

We use Proposition 3.1 to extend the above result to

PROPOSITION 3.6: *Suppose that* $1 < N = [N, G] \trianglelefteq G$ *and* $\mathrm{Z}(G) \cap N = 1$, *for some finite group* G *satisfying Hypothesis B. If* N/M *is a chief factor of* G, *for some* $M \trianglelefteq G$ *with* $M < N$, *then* $N - M$ *is a single conjugacy class in* G.

Proof: Fix a normal subgroup M of G contained in N such that N/M is a chief factor of G, i.e., such that N/M is a minimal normal subgroup of G/M. Since $[N, G]$ is N, its image $[N/M, G/M]_{G/M}$ under the natural epimorphism $e\colon G \twoheadrightarrow G/M$ is N/M. Propositions 3.1 and 3.4 now tell us that $(N/M)^{\#} = (N/M) - 1_{G/M}$ is a single G/M-conjugacy class with size $|(N/M)^{\#}| \geq 2$. Hence any G-conjugacy class $x^G \subseteq N - M$ has image $e(x^G) = (N/M)^{\#}$ in G/M.

Suppose that $N - M$ is not a single G-conjugacy class. Then there are elements $x \in N - M$ and $y \in M$ such that $(yx)^G$ and x^G are two distinct G-conjugacy classes contained in $N - M$. Since $\mathrm{Z}(G) \cap N = 1$, it follows that $(yx)^G\mathrm{Z}(G) \neq x^G\mathrm{Z}(G) = ((x^{-1})^{-1})^G\mathrm{Z}(G)$. Hypothesis B now tells us that $(yx)^G(x^{-1})^G = ((yx)x^{-1})^G = y^G$. But e sends both conjugacy classes $(yx)^G, (x^{-1})^G \subseteq N - M$ onto $(N/M)^{\#}$, and sends $y^G \subseteq M$ to $1_{G/M}$. So $(N/M)^{\#}(N/M)^{\#} = 1_{G/M}$. This is false because $|(N/M)^{\#}| \geq 2$. Thus $N - M$ is a single G-conjugacy class, and the proposition holds. ∎

Finally, we note the following easy consequence of Propositions 3.1 and 3.4.

PROPOSITION 3.7: *Any finite group* G *satisfying Hypothesis B is solvable.*

Proof: If G is not solvable, then it has two normal subgroups M and N such that $M \triangleleft N \trianglelefteq G$ and N/M is a non-abelian minimal normal subgroup of G/M. This is impossible by Proposition 3.4, because $[N/M, G/M]_{G/M}$ must be N/M, and G/M satisfies Hypothesis B by Proposition 3.1. So this proposition holds. ∎

4. Nilpotent groups

All finite abelian groups G satisfy both Hypothesis A and Hypothesis B. In this section we're going to study non-abelian nilpotent groups G satisfying one of these hypotheses. Such G are closely related to Camina p-groups.

We follow [2] by saying that a finite group G is a **Camina group** if each non-trivial coset $x[G,G] \in (G/[G,G])^{\#}$ is a single G-conjugacy class $x^G = x[x,G]$. This is obviously equivalent to

$$[x,G] = [G,G], \quad \text{for all } x \in G - [G,G]. \tag{4.1}$$

Every abelian group is a Camina group. The other nilpotent Camina groups satisfy

PROPOSITION 4.2: *Any non-abelian nilpotent Camina group G is a p-group, for some prime p.*

Proof: Because G is nilpotent and non-abelian, it has a non-abelian Sylow p-subgroup G_p, for some prime p. Furthermore, G is the direct product $G_p \times G_{p'}$ of G_p with the unique Hall p'-subgroup $G_{p'}$ of G. So $[G,G]$ is the direct product of $[G_p,G_p] \le G_p$ with $[G_{p'},G_{p'}] \le G_{p'}$.

There is some element $x \in G_p - [G_p,G_p] = G_p - [G,G]$. By (4.1) we have $[x,G] = [G,G] = [G_p,G_p] \times [G_{p'},G_{p'}]$. But $[x,G] \subseteq G_p$, since $x \in G_p \trianglelefteq G$. It follows that $[G_{p'},G_{p'}] = 1$.

The abelian direct factor $G_{p'}$ is now central in G. Hence $[y,G] = 1 < [G,G]$, for any $y \in G_{p'}^{\#} = G_{p'} - [G,G]$. But $[y,G] = [G,G]$ by (4.1). This contradiction shows that $G_{p'} = 1$. Therefore $G = G_p$ is a p-group. ∎

Camina p-groups have been studied extensively, notably in [2], [5] and [6]. As a result of those studies we have

THEOREM 4.3: *The following properties are equivalent, for any non-abelian nilpotent finite group G:*

(4.4a) *G satisfies Hypothesis A.*

(4.4b) *G satisfies Hypothesis B, and $Z(G) \le [G,G]$.*

(4.4c) *G is a Camina p-group, for some prime p.*

Proof: Suppose that G satisfies Hypothesis A. Then it satisfies the weaker Hypothesis B. Since G is not abelian, we may apply Proposition 3.2 with $K = G$ to conclude that $Z(G) \le [G,G]$. Therefore (4.4b) holds when (4.4a) does.

If G satisfies (4.4b), then Proposition 3.3 for $N = G$ tells us that (4.1) holds. So G is a Camina group. Now Proposition 4.2 completes the proof that (4.4c) holds when (4.4b) does.

Finally, suppose that G is a Camina p-group, for some prime p. The class of the p-group G is either 2 or 3 by the Main Theorem of [2]. In either case, [5, Theorem 5.2(i)] and the definition of Camina groups imply that the conjugacy classes of G are given by

$$\begin{aligned} x^G &= x\gamma_2(G) && \text{if } x \in G - \gamma_2(G), \\ &= x\gamma_3(G) && \text{if } x \in \gamma_2(G) - \gamma_3(G), \\ &= \{x\} && \text{if } x \in \gamma_3(G). \end{aligned}$$

With this information it is straightforward to verify that G satisfies Hypothesis A. So (4.4a) holds when (4.4c) does, and the theorem is proved. ∎

To state an equivalent of the above theorem for Hypothesis B we shall use Philip Hall's concept of isoclinism in [3]. Let $\bar{G}$ be the factor group $G/\mathrm{Z}(G)$ of an arbitrary finite group G. For the rest of this section $\bar{x}$ and $\bar{y}$ will denote arbitrary elements of $\bar{G}$, while x and y denote arbitrary elements of G having images $\bar{x} = x\mathrm{Z}(G)$ and $\bar{y} = y\mathrm{Z}(G)$ in $\bar{G}$. Commutation in G induces a well defined function $c = c_G\colon \bar{G} \times \bar{G} \to [G, G]$, sending $\bar{x}$ and $\bar{y}$ to

$$c(\bar{x}, \bar{y}) = [x, y] \in [G, G]. \tag{4.5}$$

This function determines many things. For example, the natural homomorphism $h = h_G\colon x \mapsto x\mathrm{Z}(G)$ of $[G, G]$ into $\bar{G} = G/\mathrm{Z}(G)$ sends $c(\bar{x}, \bar{y}) \in [G, G]$ to

$$h(c(\bar{x}, \bar{y})) = [\bar{x}, \bar{y}]_{\bar{G}} \in \bar{G}. \tag{4.6}$$

Since the group $[G, G]$ is generated by all possible commutators $c(\bar{x}, \bar{y}) = [x, y]$, this implies that the function c determines the homomorphism h.

The map $x, \bar{y} \mapsto x^{\bar{y}} = x^y$ is a well defined conjugation action of $\bar{G}$ on G, leaving $[G, G]$ invariant. Since x^y is $x[x, y]$, we have

$$x^{\bar{y}} = xc(h(x), \bar{y}) \in [G, G] \tag{4.7}$$

whenever $x \in [G, G]$. Thus the conjugation action of $\bar{G}$ on $[G, G]$ is determined by c and h, and hence by c alone, as Philip Hall remarked in [3].

He also remarked that the subgroups $\gamma_n(G)$, for $n \geq 2$, are determined by c. Indeed, $\gamma_2(G)$ is

$$\gamma_2(G) = [G, G], \tag{4.8a}$$

while $\gamma_{n+1}(G) = [\gamma_n(G), G]$, for any $n \geq 2$, is the subgroup

$$\gamma_{n+1}(G) = c(h(\gamma_n(G)), \bar{G}) \leq [G, G] \tag{4.8b}$$

generated by the elements $c(h(x), \bar{y})$, for $x \in \gamma_n(G)$ and $\bar{y} \in \bar{G}$.

Our Hypothesis B also depends only on c.

PROPOSITION 4.9: *Hypothesis B for a finite group G is equivalent to*

HYPOTHESIS 4.10: *If $\bar{x}, \bar{y} \in \bar{G}$ and $\bar{x}^{\bar{G}} \neq (\bar{y}^{-1})^{\bar{G}}$, then $c(\bar{x}, \bar{G})^{\bar{y}} c(\bar{y}, \bar{G}) = c(\bar{x}\bar{y}, \bar{G})$.*

for the factor group $\bar{G} = G/\mathrm{Z}(G)$ and the function $c\colon \bar{G} \times \bar{G} \to [G, G]$.

Proof: If $\bar{x} \in \bar{G}$ is the image $x\mathrm{Z}(G)$ of $x \in G$, then the subset $x^G\mathrm{Z}(G) \subseteq G$ is the inverse image of the conjugacy class $\bar{x}^{\bar{G}} \subseteq \bar{G}$ under the natural epimorphism of G onto $\bar{G} = G/\mathrm{Z}(G)$. Similarly, $(y^{-1})^G\mathrm{Z}(G)$ is the inverse image of $(\bar{y}^{-1})^{\bar{G}}$, whenever $y \in G$ has image $\bar{y} = y\mathrm{Z}(G)$ in $\bar{G}$. We conclude that the assumption $x^G\mathrm{Z}(G) \neq (y^{-1})^G\mathrm{Z}(G)$ in Hypothesis B is equivalent to the assumption $\bar{x}^{\bar{G}} \neq (\bar{y}^{-1})^{\bar{G}}$ in Hypothesis 4.10.

We know from (2.1) that the conclusion $x^G y^G = (xy)^G$ in Hypothesis B is equivalent to $[x, G]^y[y, G] = [xy, G]$. In view of (4.5), this last equation is equivalent to the conclusion $c(\bar{x}, \bar{G})^{\bar{y}} c(\bar{y}, \bar{G}) = c(\bar{x}\bar{y}, \bar{G})$ in Hypothesis 4.10. Thus the proposition holds. ∎

Two finite groups G and H are called **isoclinic** if there exist an isomorphism i of the factor group $\bar{G} = G/\mathrm{Z}(G)$ onto $\bar{H} = H/\mathrm{Z}(G)$, and an isomorphism j of the subgroup $[G, G]$ onto $[H, H]$, such that i and j carry the map $c_G\colon \bar{G} \times \bar{G} \to [G, G]$ to the map $c_H\colon \bar{H} \times \bar{H} \to [H, H]$, in the sense that

$$c_H(i(\bar{x}), i(\bar{y})) = j(c_G(\bar{x}, \bar{y})) \in [H, H] \tag{4.11a}$$

for all $\bar{x}, \bar{y} \in \bar{G}$. The resulting pair (i, j) is called an **isoclinism** of G onto H. It follows from (4.6) that the isomorphisms i and j must carry h_G to h_H, in the sense that

$$h_H(j(x)) = i(h_G(x)) \in \bar{H} \tag{4.11b}$$

for all $x \in [G, G]$. In view of (4.7) they also carry the conjugation action of $\bar{G}$ on $[G, G]$ to that of $\bar{H}$ on $[H, H]$, in the sense that

$$j(x)^{i(\bar{y})} = j(x^{\bar{y}}) \in [H, H] \tag{4.11c}$$

for all $x \in [G,G]$ and $\bar{y} \in \bar{G}$. Finally, it follows from (4.7) that

$$j(\gamma_n(G)) = \gamma_n(H) \tag{4.11d}$$

for all $n \geq 2$.

The word "isoclinism" also denotes the relation "G is isoclinic to H." This is obviously an equivalence relation among finite groups. Because isoclinism preserves everything appearing in Hypothesis 4.10, it is clear that a finite group G satisfies that hypothesis if and only if every finite group H isoclinic to G satisfies the same hypothesis. This and Proposition 4.9 imply

PROPOSITION 4.12: *If a finite group G satisfies Hypothesis B, then so does any finite group H isoclinic to G.*

Using one of Philip Hall's important results in [3], we can prove

THEOREM 4.13: *A finite group G is non-abelian, nilpotent, and satisfies Hypothesis B if and only if it is isoclinic to a non-abelian Camina p-group, for some prime p.*

Proof: Suppose that G is non-abelian, nilpotent, and satisfies Hypothesis B. The argument on page 135 of [3] gives us some finite group H isoclinic to G such that $\mathrm{Z}(H) \leq [H,H]$. Since G is nilpotent with some class $c \geq 2$, we have $1 = \gamma_{c+1}(G) < \gamma_c(G) \leq \gamma_2(G) = [G,G]$. In view of (4.11d) this implies that $1 = \gamma_{c+1}(H) < \gamma_c(H) \leq \gamma_2(H) = [H,H]$. So H is non-abelian and nilpotent, with the same class c as G. Because G satisfies Hypothesis B, so does H by Proposition 4.12. Therefore H is a non-abelian, nilpotent, finite group satisfying Hypothesis B with $\mathrm{Z}(H) \leq [H,H]$. By Theorem 4.3 this implies that H is a non-abelian Camina p-group, for some prime p.

On the other hand, any non-abelian Camina p-group H satisfies Hypothesis B by Theorem 4.3. In view of Proposition 4.12, so does any finite group G isoclinic to H. As above, the fact that H is non-abelian and nilpotent implies the same properties for G. Thus G is a non-abelian, nilpotent, finite group satisfying Hypothesis B, and the theorem is proved. ∎

5. Non-nilpotent groups

Throughout the many lemmas in this section G will be a non-nilpotent finite group satisfying Hypothesis B, and K will be its normal subgroup $\gamma_\infty(G)$. So

K is the smallest normal subgroup N of G such that the factor group G/N is nilpotent. Because G is solvable (see Proposition 3.7), this and (2.4) imply that

$$1 < K = [K, G] \lhd G. \tag{5.1}$$

The critical step in our argument is

LEMMA 5.2: *Suppose there is some complementary subgroup C to K in G such that $\mathrm{Z}(G) \leq C$. Then the factor group $G/\mathrm{Z}(G)$ is a Frobenius group with Frobenius kernel $K\mathrm{Z}(G)/\mathrm{Z}(G) \cong K$ and complement $C/\mathrm{Z}(G)$. Furthermore, the factor group $G/(K\mathrm{Z}(G)) \cong C/\mathrm{Z}(G)$ is non-trivial, and is either cyclic or a quaternion group of order 8.*

Proof: If $C = \mathrm{Z}(G)$, then $G = KC = K\mathrm{Z}(G)$. In view of (5.1) this implies that $1 < K = [K, K\mathrm{Z}(G)] = [K, K]$, which is impossible because $K \leq G$ is solvable by Proposition 3.7. Therefore $\mathrm{Z}(G) < C$, and $C/\mathrm{Z}(G) > 1$.

Let x be any element of $C - \mathrm{Z}(G)$. Then $x \in G = K \rtimes C$ does not lie in $K\mathrm{Z}(G) = \gamma_\infty(G)\mathrm{Z}(G)$. Since $\gamma_\infty(G)$ is the group $\gamma_{c+1}(G)$ in (2.3), there is some integer $n = 1, 2, \ldots, c$ such that $x \in \gamma_n(G)\mathrm{Z}(G) - \gamma_{n+1}(G)\mathrm{Z}(G)$. We shall apply Proposition 3.3 with $N = \gamma_n(G)\mathrm{Z}(G)$. The commutator $[N, G]$ is now $[\gamma_n(G)\mathrm{Z}(G), G] = [\gamma_n(G), G] = \gamma_{n+1}(G)$ by (2.2b). So x lies in $N - [N, G]\mathrm{Z}(G)$, and $x^G = x[N, G] = x\gamma_{n+1}(G)$ by Proposition 3.3. In particular, x^G is a union of cosets of $K = \gamma_\infty(G) \leq \gamma_{n+1}(G)$. It follows that x^G is the full inverse image of its image $(xK)^{G/K}$ in the factor group G/K. Hence $|x^G|$ is $|K| \cdot |(xK)^{G/K}|$.

The natural epimorphism of G onto G/K sends C isomorphically onto G/K, and x to xK. Hence it sends $x^C \subseteq C$ one to one onto $(xK)^{G/K}$. So $|K| \cdot |x^C| = |K| \cdot |(xK)^{G/K}| = |x^G|$. We also have $|K| \cdot |C| = |K \rtimes C| = |G|$. It follows that

$$|\mathrm{C}_G(x)| = \frac{|G|}{|x^G|} = \frac{|K| \cdot |C|}{|K| \cdot |x^C|} = \frac{|C|}{|x^C|} = |\mathrm{C}_C(x)|.$$

This can only happen when $\mathrm{C}_G(x)$ is equal to its subgroup $\mathrm{C}_C(x)$. Then $\mathrm{C}_K(x) = K \cap \mathrm{C}_G(x) \leq K \cap C = 1$. Therefore $\mathrm{C}_K(x) = 1$ for every $x \in C - \mathrm{Z}(G)$.

Since $\mathrm{Z}(G)$ is contained in the factor C of the semi-direct product $G = K \rtimes C$, the factor group $\bar{G} = G/\mathrm{Z}(G)$ is the semi-direct product $\bar{K} \rtimes \bar{C}$ of its nontrivial normal subgroup $\bar{K} = K\mathrm{Z}(G)/\mathrm{Z}(G) \cong K$ with the nontrivial complementary subgroup $\bar{C} = C/\mathrm{Z}(G)$. The above arguments imply that $\mathrm{C}_{\bar{K}}(\bar{x}) = 1$, for all $\bar{x} \in \bar{C}^\# = \bar{C} - 1$. Therefore $\bar{G}$ is a Frobenius group, with Frobenius kernel $\bar{K}$ and complement $\bar{C}$ (see [4, Satz V.8.5]).

The possible Sylow subgroups of the Frobenius complement $\bar{C}$ are well known (see [4, Hauptsatz V.8.7]). They are all cyclic except perhaps the Sylow 2-subgroup, which may be a generalized quaternion group of some order $2^n \geq 8$. Since the epimorphic image $\bar{C}$ of $G/K = G/\gamma_\infty(G)$ is nilpotent, we conclude that $\bar{C}$ is either cyclic or the direct product $\bar{Q} \times \bar{D}$ of a generalized quaternion group $\bar{Q}$ with some cyclic group $\bar{D}$ of odd order. In the former case the proof of Lemma 5.2 is finished. So we may assume that the latter case holds.

The factor group $\bar{G} = G/\mathrm{Z}(G)$ satisfies Hypothesis A by Proposition 3.1. Hence so does its factor group $\bar{G}/\bar{K}$. Thus $\bar{C} \cong \bar{G}/\bar{K}$ is a non-abelian nilpotent finite group satisfying Hypothesis A. By Theorem 4.3 this implies that $\bar{C}$ is a Camina p-group, for some prime p. Of course p must be 2, since the Sylow 2-subgroup $\bar{Q}$ of $\bar{C}$ is non-trivial. The generalized quaternion group $\bar{C} = \bar{Q}$ has an element $\bar{x}$ of order $|\bar{C}|/2$ lying in $\bar{C} - [\bar{C}, \bar{C}]$. Then $\bar{x}^{\bar{C}}$ has order 2, yet is equal to $\bar{x}[\bar{C}, \bar{C}]$ by the Camina property (4.1). We conclude that $|[\bar{C}, \bar{C}]| = 2$, so that the generalized quaternion group $\bar{C}$ is a quaternion group with order 8. Hence the lemma holds. ∎

Before we can apply the preceding lemma we must find a suitable complement C to K in G. In the minimal case such a complement is given by

LEMMA 5.3: *If K is a minimal normal subgroup of G, then there is some complementary subgroup C to K in G such that $\mathrm{Z}(G) \leq C$.*

Proof: Proposition 3.4 tells us that the minimal normal subgroup K is an elementary abelian p-group, for some prime p. The factor group $\bar{G} = G/K = G/\gamma_\infty(G)$ is nilpotent. Hence it is the direct product $\bar{G}_p \times \bar{G}_{p'}$ of its unique Sylow p-subgroup $\bar{G}_p$ and its unique Hall p'-subgroup $\bar{G}_{p'}$. It follows that the inverse image of $\bar{G}_p$ is a normal Sylow p-subgroup P of G containing K such that $P/K = \bar{G}_p$. By [4, Hauptsatz I.18.1] there is some subgroup $D \leq G$ complementary to P in $G = P \rtimes D$. Then $DK/K \cong D$ is $\bar{G}_{p'}$. So D is a nilpotent p'-subgroup of G normalizing the p-subgroup P and centralizing $P/K = \bar{G}_p$. This implies that $\mathrm{N}_P(D) = \mathrm{C}_P(D)$, and that $P = \mathrm{C}_P(D)K = \mathrm{N}_P(D)K$ (see [4, Satz I.18.6]).

If $D = 1$, then $G = P$ is a p-group, and hence is nilpotent, contrary to our assumptions. So $D > 1$. The product DK is the inverse image of $\bar{G}_{p'}$, and therefore is a normal subgroup of G. Because $K \trianglelefteq G$ is abelian, the normalizer $\mathrm{N}_K(D)$ is the normal subgroup $\mathrm{N}_K(DK) \trianglelefteq G$. If $\mathrm{N}_K(D) = K$, then $P = \mathrm{N}_P(D)K = \mathrm{N}_P(D) = \mathrm{C}_P(D)$. In this case G is the direct product $P \times D$, which is nilpotent. This contradicts our assumptions. Therefore $\mathrm{N}_K(D) = 1$.

Now P is the product $\mathrm{N}_P(D)K$ of subgroups which intersect in $\mathrm{N}_P(D) \cap K = \mathrm{N}_K(D) = 1$. So $\mathrm{N}_P(D)$ is a complement to K in P. Since $D \leq \mathrm{N}_G(D)$ is a complement to P in G, and centralizes $\mathrm{N}_P(D) = \mathrm{C}_P(D)$, it follows that $C = \mathrm{N}_G(D) = \mathrm{N}_P(D) \times D$ is a complement to K in G. Clearly C contains $\mathrm{Z}(G) \leq \mathrm{C}_G(D)$. So the lemma holds. ∎

In the general case we have

LEMMA 5.4: *The factor group $G/(K\mathrm{Z}(G))$ is non-trivial, and is either cyclic or a quaternion group with order 8.*

Proof: Since $1 < K = [K, G]$, the intersection $K \cap \mathrm{Z}(G)$ is a normal subgroup of G strictly contained in K. Hence there is some normal subgroup M of G satisfying $K \cap \mathrm{Z}(G) \leq M \lhd K$ such that K/M is a chief factor of G. It follows that $N = M\mathrm{Z}(G)$ is a normal subgroup of G containing $\mathrm{Z}(G)$ with $K \cap N = M$.

The factor group $\bar{G} = G/N$ satisfies both Hypothesis B and Hypothesis A by Proposition 3.1. The natural epimorphism e of G onto $\bar{G}$ sends K onto a minimal normal subgroup $\bar{K} = KN/N \cong K/M$ of $\bar{G}$. It follows from (5.1) that $1 < \bar{K} = [\bar{K}, \bar{G}]_{\bar{G}}$. The solvability of $\bar{G}$ then implies that $\bar{K} \lhd \bar{G}$. The factor group $\bar{G}/\bar{K}$ is an epimorphic image of the nilpotent group G/K, and hence is nilpotent. So $\bar{K}$ is precisely $\gamma_\infty(\bar{G})$. Now all the assumptions in this section are satisfied with $\bar{G}$ and $\bar{K}$ in place of G and K, respectively. In addition, $\bar{G}$ satisfies Hypothesis A, so that Proposition 3.2 gives $\mathrm{Z}(\bar{G}) \leq [\bar{K}, \bar{G}]_{\bar{G}} = \bar{K}$. This forces $\mathrm{Z}(\bar{G})$ to be 1, since $\bar{K}$ is a non-central minimal normal subgroup of $\bar{G}$.

The hypotheses of Lemma 5.3 are now satisfied with $\bar{G}$ and $\bar{K}$ in place of G and K, respectively. Hence so are the hypotheses of Lemma 5.2, with the same substitutions. That lemma tells us that the factor group $\bar{G}/\bar{K} = \bar{G}/(\bar{K}\mathrm{Z}(\bar{G}))$ is non-trivial, and is either cyclic or a quaternion group of order 8. Since $\bar{K}$ has inverse image $KN = KM\mathrm{Z}(G) = K\mathrm{Z}(G)$ in G, this implies the present lemma. ∎

Now we can find the complements we need.

LEMMA 5.5: *There is always some complementary subgroup C to K in G such that $\mathrm{Z}(G) \leq C$.*

Proof: Let $\bar{G}$ be the factor group $G/(K\mathrm{Z}(G))$, and e be the natural epimorphism of G onto $\bar{G}$. Lemma 5.4 tells us that $\bar{G}$ is not 1, and is either cyclic or a quaternion group of order 8.

Suppose that $\bar{G}$ is cyclic. Then we may choose some element $x \in G$ whose image $\bar{x} = e(x)$ is not 1 and generates $\bar{G}$. The product $C = \langle x \rangle \mathrm{Z}(G)$ is an abelian subgroup of $\mathrm{C}_G(x)$ such that $G = CK$. It follows that $[G,G] \le K = \gamma_\infty(G) \le \gamma_2(G) = [G,G]$. Hence K is precisely $[G,G]$. So the fact that $\bar{x}$ is not 1 says that x lies in $G - K\mathrm{Z}(G) = G - [G,G]\mathrm{Z}(G)$. Now Proposition 3.3 for $N = G$ tells us that $x^G = x[G,G] = xK$. Hence $|\mathrm{C}_G(x)| = |G|/|x^G| = |G|/|K|$. The equation $CK = G$ implies that $|C|/|C \cap K| = |G|/|K| = |\mathrm{C}_G(x)|$. Since C is a subgroup of $\mathrm{C}_G(x)$, we conclude that $C \cap K = 1$. Thus C is a complement to K in G containing $\mathrm{Z}(G)$, and the lemma is proved when $\bar{G}$ is cyclic.

Now suppose that $\bar{G}$ is a quaternion group with order 8. We may choose some element $\bar{x} \in \bar{G} - [\bar{G},\bar{G}]$, and some element $x \in G$ having $\bar{x}$ as its image $e(x)$ in $\bar{G}$. Then $\bar{x}$ has order 4. So we may assume that x is a 2-element, i.e., that its order is a power of 2.

The product $A = \langle x \rangle \mathrm{Z}(G)$ is an abelian subgroup of $\mathrm{C}_G(x)$. Its image $e(A) = e(\langle x \rangle)$ is $\langle \bar{x} \rangle$, which is precisely the centralizer $\mathrm{C}_{\bar{G}}(\bar{x})$ of $\bar{x}$ in the quaternion group $\bar{G}$. It follows that $e(A) = e(\mathrm{C}_G(x)) = \mathrm{C}_{\bar{G}}(\bar{x}) = \langle \bar{x} \rangle$, and hence that $AK = \mathrm{C}_G(x)K$ is the full inverse image of its image $\langle \bar{x} \rangle$ in $\bar{G} = G/(K\mathrm{Z}(G))$. Since $1 < K \le K\mathrm{Z}(G)$, this implies that $AK = \mathrm{C}_G(x)K$ is also the full inverse image of its image $AK/K = \mathrm{C}_G(x)K/K = \mathrm{C}_{G/K}(xK)$ in G/K. From this information we conclude that $\mathrm{C}_G(x)K$, $\mathrm{C}_{G/K}(xK)$ and $\mathrm{C}_{\bar{G}}(\bar{x})$ have the same index 2 in G, G/K and $\bar{G}$, respectively.

The element x lies in $G - [G,G]\mathrm{Z}(G)$, because its image $\bar{x}$ in $\bar{G} = G/(K\mathrm{Z}(G))$ lies in $\bar{G} - [\bar{G},\bar{G}]$. So Proposition 3.3 for $N = G$ tells us that $x^G = x[G,G]$. Hence x^G is a union of cosets of $K = \gamma_\infty(G) \le \gamma_2(G) = [G,G]$, and thus is the inverse image of its image $(xK)^{G/K}$ in G/K.

The order of $(xK)^{G/K}$ is the index 2 of $\mathrm{C}_{G/K}(xK)$ in G/K. Hence its inverse image x^G has order $2|K|$. But x^G also has order

$$|x^G| = \frac{|G|}{|\mathrm{C}_G(x)|} = \frac{2|\mathrm{C}_G(x)K|}{|\mathrm{C}_G(x)|} = \frac{2|K|}{|\mathrm{C}_K(x)|}.$$

Therefore $\mathrm{C}_K(x) = 1$, and $\mathrm{C}_G(x)$ is a complement to K in $\mathrm{C}_G(x)K$. Because $\mathrm{C}_G(x)K = AK$, and $A \le \mathrm{C}_G(x)$, this complement $\mathrm{C}_G(x)$ is precisely A.

Let A_2 be the Sylow 2-subgroup of the abelian group A, and S be any Sylow 2-subgroup of G containing A_2. Since AK has index 2 in G, we must have $A_2 < S$. Hence $A_2 < \mathrm{N}_S(A_2)$. So the Sylow 2-subgroup A_2 of A is not a Sylow 2-subgroup of $\mathrm{N}_G(A_2)$. This implies that $A < \mathrm{N}_G(A_2)$.

The 2-element $x \in A$ lies in A_2. Hence $\mathrm{C}_K(A_2) \le \mathrm{C}_K(x) = 1$. The normalizer $\mathrm{N}_K(A_2)$ is the centralizer $\mathrm{C}_K(A_2) = 1$, since KA_2 is the semi-direct product $K \rtimes A_2$. We conclude that $K\mathrm{N}_G(A_2) \le G$ is a semi-direct product $K \rtimes \mathrm{N}_G(A_2)$ properly containing $K \rtimes A$. But $K \rtimes A$ has index 2 in G. So $K \rtimes \mathrm{N}_G(A_2)$ must equal G, and $\mathrm{N}_G(A_2)$ is a complement C to K in G with $\mathrm{Z}(G) \leq C$. Thus the proof of Lemma 5.5 is complete. ∎

The above lemma implies that $K \cap \mathrm{Z}(G) = 1$. Actually, we can do a little better.

LEMMA 5.6: *The intersection* $[G,G] \cap \mathrm{Z}(G)$ *is* 1.

Proof: We know from Lemma 5.5 that $\mathrm{Z}(G)$ is contained in some complement C to K in G, and from Lemma 5.2 that the factor group $\bar{C} = C/\mathrm{Z}(G)$ is either cyclic or quaternion of order 8. Since $K = [K,G]$, the derived group of $G = K \rtimes C$ is $[G,G] = K \rtimes [C,C]$.

If $\bar{C}$ is cyclic, then C is abelian, $[C,C]$ is 1, and $[G,G]$ is K, which intersects $\mathrm{Z}(G)$ in 1. So the lemma holds in that case.

Now assume that $\bar{C}$ is quaternion of order 8. Let e be the natural epimorphism of C onto $\bar{C}$. The center $\bar{Z} = \mathrm{Z}(\bar{C})$ of the quaternion group $\bar{C}$ is cyclic of order 2. So its inverse image $Z = e^{-1}(\bar{Z})$ is an abelian normal subgroup of C such that $\mathrm{Z}(G) < Z \lhd C$. Since $\bar{Z}$ is the only minimal subgroup of the quaternion group $\bar{C}$, it is contained in $\langle \bar{x} \rangle$, for any $\bar{x} \in \bar{C}^{\#}$. Hence Z is contained in the abelian group $\langle x \rangle \mathrm{Z}(G)$, for any $x \in C - \mathrm{Z}(G)$. It follows that $Z \leq \mathrm{Z}(C)$. Because $C/Z \cong \bar{C}/\bar{Z}$ is elementary abelian of order 4, we conclude that C is nilpotent with class 2, and that $[C,C] \leq Z$ is the cyclic group of order 2 generated by $[x,y]$, for any $x, y \in C$ such that xZ, yZ generate C/Z.

The epimorphism $e\colon C \twoheadrightarrow \bar{C}$ sends $[C,C]$ onto $[\bar{C},\bar{C}]_{\bar{C}}$, which is precisely the subgroup $\bar{Z}$ of order 2 in the quaternion group $\bar{C}$. Since $|[C,C]| = 2 = |\bar{Z}|$, it follows that $[C,C]$ intersects the kernel $\mathrm{Z}(G)$ of e in 1. Hence the subgroup $\mathrm{Z}(G) = 1 \times \mathrm{Z}(G)$ of the semi-direct product $G = K \rtimes C$ intersects $[G,G] = K \rtimes [C,C]$ in $1 = 1 \rtimes 1$, and the lemma is proved. ∎

We can now simplify K.

LEMMA 5.7: K *is a minimal normal subgroup of* G.

Proof: Suppose this is false. Since $1 < K \trianglelefteq G$, there exist two normal subgroups L and M of G such that $M \lhd L \lhd K$, while both K/L and L/M are chief factors of G. Lemma 5.5 gives us a complementary subgroup C to K in G such that $\mathrm{Z}(G) \leq C$. Hence $\mathrm{Z}(G) \cap K = 1$. By Lemma 5.2 the factor group $G/\mathrm{Z}(G)$ is a Frobenius group, with Frobenius kernel $K\mathrm{Z}(G)/\mathrm{Z}(G) \cong K$ and complement $C/\mathrm{Z}(G)$. It follows that K, L and M are equal to $[K,G]$, $[L,G]$ and $[M,G]$, respectively. Since $\mathrm{Z}(G) \cap K = 1$, Proposition 3.6 tells us that both $K - L$ and $L - M$ are single conjugacy classes in G.

The normal subgroup $N = M\mathrm{Z}(G) \trianglelefteq G$ is now the direct product $M \times \mathrm{Z}(G)$ in the semi-direct product $G = K \rtimes C$. It follows that $\bar{G} = G/N$ is also a Frobenius group, with Frobenius kernel $\bar{K} = KN/N = (K \times \mathrm{Z}(G))/(M \times \mathrm{Z}(G)) \cong K/M$ and complement $\bar{C} = CN/N = (M \rtimes C)/(M \times \mathrm{Z}(G)) \cong C/\mathrm{Z}(G)$. Furthermore, $\bar{L} = LN/N \cong L/M$ is a minimal normal subgroup of $\bar{G}$ contained in $\bar{K}$, and $\bar{K}/\bar{L} \cong K/L$ is a chief factor of $\bar{G}$. Finally, both $\bar{K} - \bar{L}$ and $\bar{L}^{\#} = \bar{L} - 1$ are single $\bar{G}$-conjugacy classes.

The Frobenius kernel $\bar{K}$ is nilpotent (see [4, Hauptsatz V.8.7]). So the minimal normal subgroup $\bar{L}$ of $\bar{G}$ contained in $\bar{K}$ must be central in $\bar{K}$. It follows that the $\bar{G}$-conjugacy class $\bar{L}^{\#}$ is a regular $\bar{C}$-conjugacy class with order $|\bar{C}|$. Similarly, the $\bar{G}/\bar{L}$-conjugacy class $(\bar{K}/\bar{L})^{\#}$ of the Frobenius factor group $\bar{G}/\bar{L}$ has the same order $|\bar{C}|$. Thus $|\bar{K}/\bar{L}| = |\bar{L}| = |\bar{C}| + 1$. The centralizer $\mathrm{C}_{\bar{G}}(\bar{x})$ of any $\bar{x} \in \bar{K} - \bar{L}$ is equal to $\mathrm{C}_{\bar{K}}(\bar{x})$, which contains $\langle \bar{x} \rangle \bar{L} \leq \langle \bar{x} \rangle \mathrm{Z}(\bar{K})$. Since $\bar{x} \notin \bar{L}$, we have $\bar{L} < \mathrm{C}_{\bar{G}}(\bar{x}) \leq \bar{K}$. It follows that

$$|\bar{x}^{\bar{G}}| = \frac{|\bar{G}|}{|\mathrm{C}_{\bar{G}}(\bar{x})|} < \frac{|\bar{G}|}{|\bar{L}|} = |\bar{C}| \cdot |\bar{K}/\bar{L}| = |\bar{C}|^2 + |\bar{C}|.$$

But $\bar{x}^{\bar{G}}$ is $\bar{K} - \bar{L}$, which has order $|\bar{K}| - |\bar{L}| = (|\bar{C}|+1)^2 - (|\bar{C}|+1) = |\bar{C}|^2 + |\bar{C}|$. This contradicts the preceding inequality, thus proving the lemma. ∎

Recall from the introduction that $F^+ \rtimes F^\times$ denotes the semi-direct product of the additive group F^+ of some finite field F with the multiplicative group $F^\times$ of F. This semi-direct product is a Frobenius group with Frobenius kernel $F^+ \rtimes 1 \cong F^+$ unless F has order $|F| = 2$. We denote by E_9 an elementary abelian group of order 9. Its automorphism group has a unique quaternion subgroup Q_8 of order 8. The corresponding semi-direct product $E_9 \rtimes Q_8$ is also a Frobenius group. With this notation, the conclusions of this section for groups satisfying Hypothesis A can be stated as

THEOREM 5.8: *The following properties are equivalent, for any non-nilpotent finite group G:*

(5.9a) *G satisfies Hypothesis A.*

(5.9b) *G satisfies Hypothesis B, and $\mathrm{Z}(G) = 1$.*

(5.9c) *G satisfies Hypothesis B, and $\mathrm{Z}(G) \leq [G, G]$.*

(5.9d) *G is isomorphic to one of the groups on the following list:*

(5.10a) *$F^+ \rtimes F^\times$, for some finite field F with $|F| > 2$.*

(5.10b) *$E_9 \rtimes Q_8$.*

Proof: Suppose that Hypothesis A holds. Then so does the weaker Hypothesis B. So all the results in this section are valid for G and its subgroup $K = \gamma_\infty(G)$.

Since $1 < K = [K, G]$, we know from Proposition 3.2 that $\mathrm{Z}(G) \leq K$. But $\mathrm{Z}(G) \cap K = 1$ by Lemma 5.5. Hence $\mathrm{Z}(G) = 1$. On the other hand, if $\mathrm{Z}(G)$ is 1, then Hypothesis A is equivalent to Hypothesis B. Thus (5.9a) holds if and only if (5.9b) does.

If G satisfies Hypothesis B, then $[G, G] \cap \mathrm{Z}(G) = 1$ by Lemma 5.6. It follows that (5.9b) is equivalent to (5.9c).

Now assume that G satisfies (5.9b). Lemma 5.7 says that $K = \gamma_\infty(G)$ is a minimal normal subgroup of G. Since $K = [K, G]$, Proposition 3.4 tells us that K is an elementary abelian p-group, for some prime p, and that $K^\# = K - 1$ is a single G-conjugacy class with size $|K^\#| \geq 2$. Lemma 5.5 gives us a complement C to K in G. Since $\mathrm{Z}(G) = 1$, Lemma 5.2 says that $G = K \rtimes C$ is a Frobenius group with Frobenius kernel K and complement C. Furthermore, it tells us that C is either cyclic or a quarternion group of order 8. The G-conjugacy class $K^\#$ in the abelian Frobenius kernel K must be a regular C-orbit of length $|C|$. Thus $|K| = |C| + 1$.

If C is cyclic, then G is the semi-direct product of an elementary abelian group K of order $p^n > 2$, for some $n \geq 1$, with a cyclic group C of order $p^n - 1$ acting faithfully on K. It follows that it is isomorphic to the group $F^+ \rtimes F^\times$ in (5.10a) for the finite field F of order p^n.

If C is a quaternion group of order 8, then $|K| = |C| + 1 = 9$. In this case G is isomorphic to the group $E_9 \rtimes Q_8$ in (5.10a). Thus (5.9b) implies (5.9d).

Suppose that $G = F^+ \rtimes F^\times$, for some finite field F with order $|F| > 2$. It is straightforward to compute that

$$\begin{aligned} x^G &= x[G,G] = x(F^+ \times 1) && \text{if } x \in G - [G,G], \\ &= [G,G]^\# = (F^+)^\# \times 1 && \text{if } x \in [G,G] - 1, \\ &= 1 && \text{if } x = 1, \end{aligned}$$

for any $x \in G$. Given this information, it is easy to verify that Hypothesis A holds.

Suppose that $G = E_9 \rtimes Q_8$. Then we have

$$\begin{aligned} x^G &= x\gamma_2(G) = x(E_9 \rtimes \mathrm{Z}(Q_8)) && \text{if } x \in G - \gamma_2(G), \\ &= x\gamma_3(G) = x(E_9 \times 1) && \text{if } x \in \gamma_2(G) - \gamma_3(G), \\ &= \gamma_3(G)^\# = (E_9)^\# \times 1 && \text{if } x \in \gamma_3(G) - 1, \\ &= 1 && \text{if } x = 1, \end{aligned}$$

for any $x \in G$. Again it is easy to verify that G satisfies Hypothesis A. Thus (5.9d) implies (5.9a), and the theorem is proved. ∎

Our conclusions for groups satisfying Hypothesis B are given in

THEOREM 5.11: *A non-nilpotent finite group G satisfies Hypothesis B if and only if it is isoclinic to some group on the list (5.10).*

Proof: If G satisfies Hypothesis B, then all the results in this section hold for G and its subgroup $K = \gamma_\infty(G)$. In particular, Lemma 5.6 tells us that $[G,G] \cap \mathrm{Z}(G) = 1$. By the argument on page 134 of [3], this implies that G is isoclinic to its factor group $\bar{G} = G/\mathrm{Z}(G)$. This factor group satisfies Hypothesis A by Proposition 3.1. Furthermore, $\gamma_\infty(\bar{G})$ is the image $\bar{K} = K\mathrm{Z}(G)/\mathrm{Z}(G) \cong K > 1$ of $\gamma_\infty(G) = K$. So Theorem 5.8 tells us that $\bar{G}$ is isomorphic to one of the groups on the list (5.10). Thus G is isoclinic to such a group.

Any group H on the list (5.10) is non-nilpotent and satisfies Hypothesis B by Theorem 5.8. In view of (4.11d) and Proposition 4.12, so does any finite group G isoclinic to H. Thus the theorem holds. ∎

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